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Montroll–Weiss Problem, Fractional Equations, and Stable Distributions

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Asymptotic solutions of the m-dimensional Montroll–Weiss' jump problem are obtained. They cover both the subdiffusive and the superdiffusive regime, obey fractional differential equations, and are expressed in terms of stable distributions. Analytical investigation and numerical calculations of anomalous diffusion distributions are performed and their properties are discussed.

1. INTRODUCTION

The Montroll–Weiss (MW) problem [17] is formulated as the problem of finding the probability distribution $p(x, t)$, $x \in \mathrm{R}^m$, for a particle performing random instantaneous jumps $R_1, R_2, \ldots, R_j, \ldots \in \mathrm{R}^m$ at random waiting times $T_1, T_1 + T_2, \ldots, T_1 + T_2 + T_j, \ldots, T_i \in \mathrm{R}^1_+$. The random variables R_i and T_i are independent and their distribution densities $p(x)$ and $q(t)$ do not depend on time and place, respectively. Numerous examples of applications of the model to concrete physical and biological systems and processes are reviewed in refs. 3, 12, and 28. We concentrate only on mathematical aspects of this problem.

The Fourier–Laplace transform of the distribution

$$p(k, \lambda) = \int_{\mathrm{R}^m} dx \int_0^\infty dt\, e^{ik\cdot x - \lambda t}\, p(x, t), \qquad k \in \mathrm{R}^m$$

is easily expressed in terms of the Laplace transform of the waiting-time distribution

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$$q(\lambda) = \int_0^\infty e^{-\lambda t} q(t)\, dt$$

and the Fourier transform of the jump distribution

$$p(k) = \int_{\mathrm{R}^m} e^{ik\cdot x} p(x)\, dx$$

If the particle begins its history waiting at the origin of space-time coordinates ($x = 0$, $t = 0$), then

$$p(k, \lambda) = \frac{1 - q(\lambda)}{\lambda[1 - p(k)q(\lambda)]} \tag{1.1}$$

This is just the Montroll–Weiss result.

We will suppose the jump distribution $p(x)$ to be isotropic, so $p(k)$ is a function of $|k|$ only.

At large time when the particle has performed many jumps and the spatial distribution of probability becomes wide, the density

$$p(x, t) = i^{-1}(2\pi)^{-m-1} \int_{\mathrm{R}^m} dk \int_L d\lambda\, e^{ik\cdot x + \lambda t} p(k, \lambda)$$

is determined by the behavior of the transform $p(k, \lambda)$ in the region of small k and λ.

If

$$\int_{\mathrm{R}^m} p(x)|x|^2\, dx \equiv \langle R^2 \rangle \tag{1.2}$$

and

$$\int_0^\infty q(t) t\, dt \equiv \langle T \rangle \tag{1.3}$$

are finite, then

$$1 - q(\lambda) \sim \langle T \rangle \lambda, \qquad \lambda \to 0 \tag{1.4}$$

and

$$p(k) \sim 1 - \langle R^2/2 \rangle |k|^2, \qquad k \to 0 \tag{1.5}$$

On substituting (1.4) and (1.5) into Eq. (1.1), we obtain

$$p(k, \lambda) \sim p^{\mathrm{as}}(k, \lambda) = \frac{1}{\lambda + Dk^2}, \qquad \lambda \to 0, \quad k \to 0$$

where

$$D = \langle R^2/2 \rangle / \langle T \rangle$$

This is nothing but the Fourier–Laplace transform of the Gauss distribution

$$p^{\mathrm{as}}(x, t) = (4\pi D t)^{-m/2} \exp\{-x^2/(4Dt)\} \tag{1.6}$$

obeying the ordinary diffusion equation

$$\frac{\partial p^{\mathrm{as}}(x, t)}{\partial t} = D \Delta p^{\mathrm{as}}(x, t) + \delta(x)\delta(t) \tag{1.7}$$

where $\delta(x)$ and $\delta(t)$ are m-dimensional and one-dimensional Dirac functions, respectively. We will call Eq. (1.6) the ordinary diffusion distribution (ODD). The width of the ODD increases in the fashion $t^{1/2}$.

If one or both of values (1.2) and (1.3) are infinite but the corresponding conditions

$$\int_{|x|>r} p(x)\, dx \sim A r^{-\alpha}, \qquad r \to \infty, \quad 0 < \alpha < 2 \tag{1.8}$$

and

$$\int_t^{\infty} q(\tau)\, d\tau \sim B t^{-\beta}, \qquad t \to \infty, \quad 0 < \beta < 1 \tag{1.9}$$

hold, we obtain a model of anomalous diffusion, for which a complete set of solutions has not been found. The aim of this article is to fill the gap.

2. STABLE DISTRIBUTIONS

Let us recall some facts from the stable law theory. A spherically symmetric, m-dimensional stable density with the characteristic exponent α, $g_m^{\alpha}(x)$, is defined by its characteristic function

$$\mathfrak{g}_m^{(\alpha)}(k) \equiv \int_{\mathrm{R}^m} e^{ik\cdot x} g_m^{(\alpha)}(x)\, dx = e^{-|k|^{\alpha}}, \qquad 0 < \alpha \leq 2 \tag{2.1}$$

The backward transformation of (2.1) can be performed in terms of elementary functions in two cases only:

$$g_m^{(2)}(x) = (4\pi)^{-m/2} \exp\{-|x|^2/4\}$$

which is the Gauss distribution, and

$$g_m^{(1)}(x) = \Gamma((m+1)/2)[\pi(1 + |x|^2)]^{-(m+1)/2}$$

which is the Cauchy distribution. The rest of the densities are expressed in terms of Bessel functions [33]

$$g_m^{(\alpha)}(x) = (2\pi)^{-m/2} \int_0^\infty e^{-s^\alpha} J_{m/2-1}(s|x|)(s|x|)^{1-m/2} s^{m-1}\, ds$$

With this integral representation, the two following series are useful:

$$g_m^{(\alpha)}(x) = \frac{\alpha}{2}(4\pi)^{-m/2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma((\alpha n + m)/2)}{\Gamma(n)\Gamma(1-\alpha n/2)} \left(\frac{|x|}{2}\right)^{-m-n\alpha}$$

$$g_m^{(\alpha)}(x) = \frac{2}{\alpha}(4\pi)^{-m/2} \sum_{n=0}^{\infty} (-1)^{n} \frac{\Gamma((2n + m)/\alpha)}{\Gamma(n+m/2)\Gamma(n+1)} \left(\frac{|x|}{2}\right)^{2n}$$

The first converges for $\alpha \in (0, 1)$ and is asymptotic for $\alpha \in [1, 2)$, the second on the contrary, converges in the range $\alpha \in [1, 2]$ and is asymptotic for $\alpha \in (0, 1)$.

The main reason stable laws arise in the diffusion problem is that they play the same limit role by summing independent random variables with an infinite variance as the Gauss law in the case of a finite variance.

If m-dimensional independent vectors $R_1, \ldots, R_n$ have common spherically symmetric density $p(x)$ obeying condition (1.8), then a sequence of positive numbers a_n can be found such that the normalized sum

$$Z_n = X_n/a_n, \qquad X_n = \sum_{i=1}^{n} R_i$$

will be distributed according to the m-dimensional symmetric, stable density $g_m^{(\alpha)}(x)$ as $n \to \infty$:

$$\text{Prob}\{Z_n \in dx\} \to g_m^{(\alpha)}(x)\, dx, \qquad n \to \infty \tag{2.2}$$

The sequence can be chosen in the form

$$a_n = a_1^{(m)}(\alpha) n^{1/\alpha}$$

where $a_1^{(m)}(\alpha)$ is calculated in the Appendix.

The family of stable laws is not exhausted by the symmetric distributions [26]. For example, the family of standardized one-dimensional stable laws is a two-parameter set of distributions $g_1^{(\alpha,\theta)}(t)$. The parameter θ characterizes the degree of asymmetry: if $\theta = 0$, then the distribution is symmetric. If $\alpha < 1$ and $\theta = 1$, then the distributions are concentrated on the positive semiaxis only (so-called one-sided stable distributions), One of them is known as the Smirnov (or Lévy) distribution:

$$g_1^{(1/2,1)}(t) = \frac{1}{2\sqrt{\pi}} t^{-3/2} e^{-1/(4t)}, \qquad t > 0$$

The other one-sided distributions are not expressible in terms of elementary functions, but their Laplace transforms

$$\tilde{g}_1^{(\beta,1)}(\lambda) \equiv \int_0^\infty e^{-\lambda t} g_1^{(\beta,1)}(t)\, dt = e^{-\lambda^\beta}, \qquad t > 0 \tag{2.3}$$

and Mellin transforms

$$\bar{g}_1^{(\beta,1)}(s) \equiv \int_0^\infty t^s g^{(\beta,1)}(t)\, dt = \Gamma(1 - s/\beta)/\Gamma(1 - s)$$

are written in a simple form. The following relation is easily proved with the use of the preceding equality:

$$\int_0^\infty g_1^{(2)}(rt^{\beta/2}) g_1^{(\beta,1)}(t) t^{\beta/2}\, dt = \beta^{-1} r^{-1-2/\beta} g_1^{(\beta/2,1)}(r^{-2/\beta}) \tag{2.4}$$

The density $g_1^{(\beta,1)}(t)$ can be represented in the form of a convergent series for any $t > 0$,

$$g_1^{(\beta,1)}(t) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(1 + n\beta) \sin(n\pi\beta) t^{-n\beta-1}$$

There exists also an asymptotic series as $t \to 0$ [26], the leading term of which has the form

$$g_1^{(\beta,1)}(t) \sim at^{-\gamma} \exp\{-bt^\delta\} \tag{2.5}$$

where

$$a = [2\pi(1 - \beta)]^{-1/\beta} \beta^{1/(2-2\beta)}$$
$$\gamma = (1 - \beta/2)/(1 - \beta)$$
$$b = (1 - \beta)\beta^\delta$$
$$\delta = \beta/(1 - \beta)$$

If independent variables $T_1, \ldots, T_n \in \mathrm{R}_+^1$ have a common distribution density $q(t)$ satisfying condition (1.9), then a sequence of positive numbers b_n exists such that the normalized sum

$$\Theta_n = \sum_{i=1}^{n} T_i/b_n$$

is distributed according to the one-sided stable law $g_1^{(\beta,1)}(t)$ as $n \to \infty$:

$$\mathrm{Prob}\{t \le \Theta_n < t + dt\} \to g_1^{(\beta,1)}(t)\, dt, \qquad n \to \infty \tag{2.6}$$

It is known [26] that b_n can be chosen in the form

$$b_n = b_1(\beta) n^{1/\beta}$$

with

$$b_1(\beta) = [B\Gamma(1-\beta)]^{1/\beta}$$

3. ASYMPTOTIC SOLUTIONS OF MW PROBLEM

Let us return to the MW problem.

According to the Tauberian theorem (see, for example, ref. 8) the asymptotic expressions (1.8) and (1.9) provide the following behavior of transforms $p(k)$ and $q(\lambda)$ at small arguments:

$$1 - p(k) \sim A'|k|^{\alpha}, \quad k \to 0, \qquad A' = 2^{-\alpha} A \frac{\Gamma(m/2)\Gamma(1-\alpha/2)}{\Gamma((\alpha+m)/2)}$$

$$1 - q(\lambda) \sim B'\lambda^{\beta}, \quad \lambda \to 0, \qquad B' = \Gamma(1-\beta)B$$

Three cases arise in addition to the normal case considered above:

A. $1 - p(k) \sim A'|k|^{\alpha}$, $1 - q(\lambda) \sim \langle T\rangle\lambda$.
B. $1 - p(k) \sim \langle R^2/2\rangle|k|^2$, $1 - q(\lambda) \sim B'\lambda^{\beta}$.
C. $1 - p(k) \sim A'|k|^{\alpha}$, $1 - q(\lambda) \sim B'\lambda^{\beta}$.

Substituting these expressions in Eq. (1.1) we obtain respectively

$$p_{\mathrm{A}}^{\mathrm{as}}(k, \lambda) = \frac{1}{\lambda + D_{\mathrm{A}}|k|^{\alpha}}, \qquad D_{\mathrm{A}} = \frac{A'}{\langle T\rangle} \tag{3.1}$$

$$p_{\mathrm{B}}^{\mathrm{as}}(k, \lambda) = \frac{\lambda^{\beta-1}}{\lambda^{\beta} + D_{\mathrm{B}}|k|^{2}}, \qquad D_{\mathrm{B}} = \frac{\langle R^2/2\rangle}{B'} \tag{3.2}$$

$$p_{\mathrm{C}}^{\mathrm{as}}(k, \lambda) = \frac{\lambda^{\beta-1}}{\lambda^{\beta} + D_{\mathrm{C}}|k|^{\alpha}}, \qquad D_{\mathrm{C}} = \frac{A'}{B'} \tag{3.3}$$

On reversing the Laplace transformation in the case (3.1)

$$p_{\mathrm{A}}^{\mathrm{as}}(k, t) = \frac{1}{2\pi i}\int_C e^{\lambda t}\frac{d\lambda}{\lambda + D_{\mathrm{A}}|k|^{\alpha}} = \exp\{-D_{\mathrm{A}}|k|^{\alpha}t\}$$

we readily arrive at the characteristic function (2.1) of the m-dimensional, spherically symmetric stable distribution with the characteristic exponent α describing the subdiffusion behavior

$$p_{\mathrm{A}}^{\mathrm{as}}(x, t) = (D_{\mathrm{A}}t)^{-m/\alpha} g_m^{(\alpha)}(x(D_{\mathrm{A}}t)^{-1/\alpha}) \tag{3.4}$$

The variance of the distribution diverges and cannot be used for description of the width of the diffusion packet. But it is clear from Eq. (3.4) that the width grows with t by the law $t^{1/\alpha}$, $\alpha < 2$, i.e., faster then in the normal case. Therefore we observe superdiffusive behavior.

The two other transforms (3.2) and (3.3) can be represented in the same form,

$$p^{\mathrm{as}}(k, \lambda) = \lambda^{\beta-1} \int_0^\infty \exp\{-[\lambda^\beta + D|k|^\alpha]y\}\, dy$$

On reversing the Laplace transformation

$$p^{\mathrm{as}}(k, t) = \int_0^\infty dy\, e^{-D|k|^\alpha y}(2\pi i)^{-1} \int_L d\lambda\, \lambda^{\beta-1} \exp\{\lambda t - \lambda^\beta y\}$$

taking the inside integral by parts

$$\int_L d\lambda\, \lambda^{\beta-1} e^{\lambda t - \lambda^\beta y} = -(\beta y)^{-1} \int_L e^{\lambda t} de^{-\lambda^\beta y} = t(\beta y)^{-1} \int_L e^{-\lambda^\beta y + \lambda t}\, d\lambda$$

and making the change of variable

$$s = y^{1/\beta}\lambda$$

we obtain

$$p^{\mathrm{as}}(k, t) = \beta^{-1} t \int_0^\infty dy\, e^{-D|k|^\alpha y} y^{-1-1/\beta} \left[(2\pi i)^{-1} \int_S e^{sy^{-1/\beta} t - s^\beta}\, ds \right]$$

As is clear from Eq. (2.3), the square brackets contain the one-sided density $g_1^{(\beta,1)}(ty^{-1/\beta})$, so the expression can be rewritten in the following way:

$$p^{\mathrm{as}}(k, t) = \int_0^\infty \exp\{-D|k|^\alpha t^\beta / \tau^\beta\} g_1^{(\beta,1)}(\tau)\, d\tau$$

The inverse Fourier transformation leads to the final result:

$$p^{\mathrm{as}}(x, t) = (Dt^\beta)^{-m/\alpha} \Psi_m^{(\alpha,\beta)}(|x|(Dt^\beta)^{-1/\alpha}) \tag{3.5}$$

where

$$\Psi_m^{(\alpha,\beta)}(r) = \int_0^\infty g_m^{(\alpha)}(r\tau^{\beta/\alpha}) g_1^{(\beta,1)}(\tau) \tau^{m\beta/\alpha} d\tau \tag{3.6}$$

and

$$D = \begin{cases} D_{\mathrm{B}} & \text{if} \quad \alpha = 2 \\ D_{\mathrm{C}} & \text{if} \quad \alpha < 2 \end{cases}$$

As one can see from Eq. (3.5), the law of the diffusion packet spreading is determined by the ratio β/α: the process has superdiffusive behavior if $\beta > \alpha/2$ and subdiffusive behavior if $\beta < \alpha/2$. We will call (3.6) the anomalous diffusion distribution (ADD). When $\beta = \alpha/2$ the width of the diffusion packet grows in time as in the normal case, but its form differs from Gaussian and depends on α.

The ADDs can be obtained in a simple way as a result of (2.2) and (2.6). Let $N(t)$ be a random number of jumps in time t. The position of the particle at the time t is

$$X(t) = \sum_{i=1}^{N(t)} R_i$$

According to (2.2), the conditional probability is given by

$$P\{X(t) \in dx \,|\, N(t) = n\} \sim g_m^{(\alpha)}(x/a_n)a_n^{-m}\, dx \tag{3.7}$$

with

$$a_n = a_1^{(m)}(\alpha)n^{1/\alpha}$$

If $\langle T\rangle < \infty$, then $N \sim t/\langle T\rangle$ as $t \to \infty$ and we arrive at Eq. (3.4). If $\langle T\rangle = \infty$, but the condition (1.9) holds, then the probability

$$W_n \equiv \mathrm{Prob}\{N(t) = n\} = \mathrm{Prob}\left\{\sum_{i=1}^{n} T_i < t\right\} - \mathrm{Prob}\left\{\sum_{i=1}^{n+1} T_i < t\right\}$$

is expressed in terms of densities $g_1^{(\beta,1)}$ as follows:

$$W_n \sim n\beta^{-1}g_1^{(\beta,1)}(t/b_n)t/b_n, \qquad t \to \infty$$

with

$$b_n = b_1(\beta)n^{1/\beta}$$

Now, averaging Eq. (3.7) over all possible numbers of jumps

$$p(x, t) = \sum_{n=1}^{\infty} g_m^{(\alpha)}(x/a_n)a_n^{-m}W_n$$

and passing from summation to integration with respect to $\tau = t/b_n$, we readily get Eqs. (3.5) and (3.6).

4. FRACTIONAL DIFFUSION EQUATIONS

There exist different constructions of fractional derivatives [15, 19, 20]. We recall here only two which will be needed below. The first is the Riemann–Liouville derivative

$$D^{\mu}_{0+} f(t) = \frac{1}{\Gamma(1 - \mu)} \frac{d}{dt} \int_0^t \frac{f(\tau)\, d\tau}{(t - \tau)^{\mu}}, \qquad \mu < 1$$

Because the integral is nothing but the convolution of the functions $f(t)$ and

$$t_+^{-\mu} = \begin{cases} t^{-\mu}, & t > 0 \\ 0, & t < 0 \end{cases}$$

i.e.,

$$h(t) \equiv \int_0^t (t - \tau)^{-\mu} f(\tau)\, d\tau = f(t) * t_+^{-\mu}, \qquad t > 0$$

its Laplace transform has the form

$$h(\lambda) \equiv \int_0^{\infty} e^{-\lambda t} h(t)\, dt = f(\lambda) \int_0^{\infty} t^{-\mu} e^{-\lambda t}\, dt = \lambda^{\mu-1} f(\lambda) \Gamma(1 - \mu)$$

It is easy to see now that

$$\int_0^{\infty} e^{-\lambda t} D^{\mu}_{0+} f(t)\, dt = \lambda^{\mu} f(\lambda) \tag{4.1}$$

The second kind of fractional derivative we will use below is given by the m-dimensional Riesz operator,

$$(-\Delta)^{\nu/2} f(x) = \frac{1}{d_{m,l}(\nu)} \int_{\mathrm{R}^m} \frac{\Delta^l_y f(x)}{|y|^{m+\nu}}\, dy$$

where $l > \alpha$, $x \in \mathrm{R}^m$, $y \in \mathrm{R}^m$,

$$\Delta^l_y f(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x - ky)$$

and

$$d_{m,l}(\nu) = \int_{\mathrm{R}^m} (1 - e^{iy_1})^l |y|^{-m-\nu}\, dy$$

One can show [20]

$$\int_{\mathrm{R}^m} e^{ikm} (-\Delta)^{\nu/2} f(x)\, dx = |k|^{\nu} f(k) \tag{4.2}$$

where $f(k)$ is the Fourier transform of the function $f(x)$. The formula generalizes the relation

$$\int_{\mathrm{R}^m} e^{ikx} \Delta f(x)\, dx = -k^2 f(k)$$

where Δ is the m-dimensional Laplacian.

Now, if we rewrite Eq. (3.1) in the form

$$\lambda p^{\mathrm{as}}(k, \lambda) = -D|k|^{\alpha} p^{\mathrm{as}}(x, t) + 1$$

and invert the Fourier–Laplace transform using the relation (4.2), then we obtain the fractional superdiffusion equation

$$\frac{\partial p^{\mathrm{as}}(x, t)}{\partial t} = -D(-\Delta)^{\alpha/2} p^{\mathrm{as}}(x, t) + \delta(x)\delta(t) \tag{4.3}$$

Applying such a procedure to Eqs. (3.2) and (3.3) with the use of Eq. (4.1) we get the equation

$$D_{0+}^{\beta} p^{\mathrm{as}}(x, t) = -D(-\Delta)^{\alpha/2} p^{\mathrm{as}}(x, t) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\, \delta(x) \tag{4.4}$$

which gives the fractional subdiffusive equation in the case $\alpha = 2$:

$$D_{0+}^{\beta} p^{\mathrm{as}}(x, t) = D\Delta p^{\mathrm{as}}(x, t) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\, \delta(x) \tag{4.5}$$

Writing (4.4) in a more general form,

$$D_{0+}^{\beta-\gamma} p^{\mathrm{as}}(x, t) = -DD_{0+}^{-\gamma}(-\Delta)^{\alpha/2} p^{\mathrm{as}}(x, t) + \frac{t^{-\beta+\gamma}}{\Gamma(1-\beta+\gamma)}\, \delta(x) \tag{4.6}$$

and setting here $\gamma = \beta - 1$ or $\gamma = \beta$, we obtain two more special forms of the equation:

$$\frac{\partial p^{\mathrm{as}}(x, t)}{\partial t} = -DD_{0+}^{1-\beta}(-\Delta)^{\alpha/2} p^{\mathrm{as}}(x, t) + \Delta(x)\delta(t) \tag{4.7}$$

$$p^{\mathrm{as}}(x, t) = -DD_{0+}^{-\beta}(-\Delta)^{\alpha/2} p^{\mathrm{as}}(x, t) + \delta(x) \tag{4.8}$$

According to ref. 20,

$$\begin{aligned} D_{0+}^{-\beta} f(t) &= \frac{1}{\Gamma(1+\beta)} \frac{d}{dt} \int_0^t f(\tau)(t-\tau)^{\beta}\, d\tau \\ &= \frac{1}{\Gamma(\beta)} \int_0^t f(\tau)(t-\tau)^{\beta-1}\, d\tau \\ &= I_{0+}^{\beta} f(t) \end{aligned}$$

is a fractional integral of order β.

Equations (4.6)–(4.8) generalize the ordinary diffusion equation (1.7) to the case of anomalous diffusion.

5. ANOMALOUS DIFFUSION DISTRIBUTIONS

The following properties of the obtained ADDs can be more or less easily established via the relations given in Section 2 and some simple arguments.

1. The densities $\Psi_{m+2}^{(\alpha,\beta)}(r)$ and $\Psi_m^{(\alpha,\beta)}(r)$ are linked via the relation

$$\Psi_{m+2}^{(\alpha,\beta)}(r) = -\frac{1}{2\pi r}\frac{d\Psi_m^{(\alpha,\beta)}(r)}{dr}$$

2. Similarly to the normal case, the projection of a diffusible m-dimensional vector $X(t)$ on an m'-dimensional subspace ($m' < m$) diffuses according to an m'-dimensional law with the same parameters α and β.

3. In contrast to the normal case, different coordinates $X_1(t), \ldots, X_m(t)$ of a particle performing anomalous diffusion are not independent of each other.

4. The ADD $\Psi_m^{(\alpha,\beta)}(r)$ is a decreasing function of r and its maximal value $\Psi_m^{(\alpha,\beta)}(0)$ is finite only if $m < \alpha$:

$$\Psi_m^{(\alpha,\beta)}(0) = \frac{\Gamma(1 + m/\alpha)\Gamma(1 - m/\alpha)}{(4\pi)^{m/2}\Gamma(1 + m/2)\Gamma(1 - m\beta/\alpha)}$$

In particular,

$$\Psi_1^{(\alpha,\beta)}(0) = \frac{\csc(\pi/\alpha)}{\alpha\Gamma(1 - \beta/\alpha)}$$

5. In the case $\beta = 1$, the ADD becomes the stable distribution:

$$\Psi_m^{(\alpha,1)}(r) = g_m^{(\alpha)}(r)$$

6. If $\alpha = 2$ and $\beta < 1$, then

$$\Psi_1^{(2,\beta)}(0) = [2\Gamma(1 - \beta/2)]^{-1}$$

$$\Psi_2^{(2,\beta)}(r) \sim [2\pi\Gamma(1 - \beta)]^{-1}|\ln r|, \qquad r \to 0$$

and for $m \geq 3$

$$\Psi_m^{(2,\beta)}(r) \sim (4\pi)^{-m/2}[\Gamma(m/2 - 1)/\Gamma(1 - \beta)](r/2)^{-(m-2)}, \qquad r \to 0$$

7. At large distances

$$\Psi_m^{(2,\beta)}(r) \sim (4\pi)^{-m/2}(2 - \beta)^{-1/2}\beta^{[(m+1)\beta/2-1]/(2-\beta)} \times (r/2)^{-m(1-\beta)/(2-\beta)} \exp\{-(2 - \beta)\beta^{\beta/(2-\beta)}(r/2)^{2/(2-\beta)}\} \qquad (5.1)$$

8. In the one-dimensional case with $\alpha = 2$ the expression (3.6) is essentially simplified due to (2.4):

$$\Psi_1^{(2,\beta)}(r) = \beta^{-1} r^{-1-2/\beta} g_1^{(\beta/2,1)}(r^{-2/\beta}) \tag{5.2}$$

Setting $\beta = 1$, we obtain

$$\Psi_1^{(2,1)}(r) = \frac{1}{\sqrt{4\pi}} e^{-r^2/4}$$

and in the case $\beta = 2/3$

$$\Psi_1^{(2,2/3)}(r) = \frac{1}{2\pi} \sqrt{r} K_{1/3}(2r^{3/2}/\sqrt{27})$$

9. In the case $\alpha = 1$, $\beta = 1/2$ the ADDs of all dimensions are expressed in terms of the incomplete gamma function:

$$\Psi_m^{(1,1/2)}(r) = \frac{2}{\sqrt{\pi}} \frac{\Gamma((m+1)/2)}{(4\pi)^{(m+1)/2}} e^{r^2/4} \Gamma(1 - (m+1)/2, r^2/4)$$

For odd dimensions

$$\Psi_m^{(1,1/2)}(r) = \frac{2}{\sqrt{\pi}} \frac{\Gamma((m+1)/2)}{(4\pi)^{(m+1)/2}} \left(\frac{r^2}{4}\right)^{\mu} e^{r^2/4} E_{(m+1)/2}(r^2/4)$$

where $\mu = 1 - (m+1)/2$.

10. The Mellin transform of the ADD is of the form

$$\begin{aligned} \tilde{\Psi}_m^{(\alpha,\beta)}(s) &\equiv \int_0^\infty \Psi_m^{(\alpha,\beta)}(r) r^{s-1}\, dr \\ &= \frac{2^s \Gamma(1 - (m-s)/\alpha) \Gamma(s/2) \Gamma((m-s)/\alpha)}{\alpha (4\pi)^{m/2} \Gamma(1 - (m-s)\beta/\alpha) \Gamma((m-s)/2)} \end{aligned}$$

11. When $\alpha = 2$

$$\langle |X(t)|^{2n} \rangle = \frac{\Gamma(n + m/2)\Gamma(n+1)}{\Gamma(m/2)\Gamma(n\beta + 1)} (4Dt^{\beta})^n \tag{5.3}$$

One-sided stable densities $g_1^{(\beta,1)}(t)$ and superdiffusion distributions $\Psi_m^{(\alpha,1)}(r)$, being merely the spherical symmetric stable densities

$$\Psi_m^{(\alpha,1)}(r) = g_m^{(\alpha)}(r)$$

can be found in ref. 26. Subdiffusive distributions $\Psi_m^{(2,\beta)}(r)$ for $m = 1, 2,$ and 3 are plotted in Figs. 1–3.

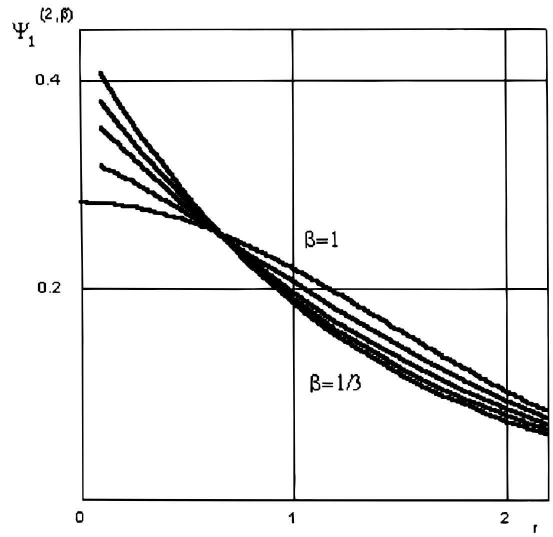

Fig. 1. One-dimensional anomalous diffusion distributions $\Psi_1^{(2,\beta)}(r)$ for $\beta = 1/3$, 1/2, 2/3, 5/6, and 1 (the normal distribution).

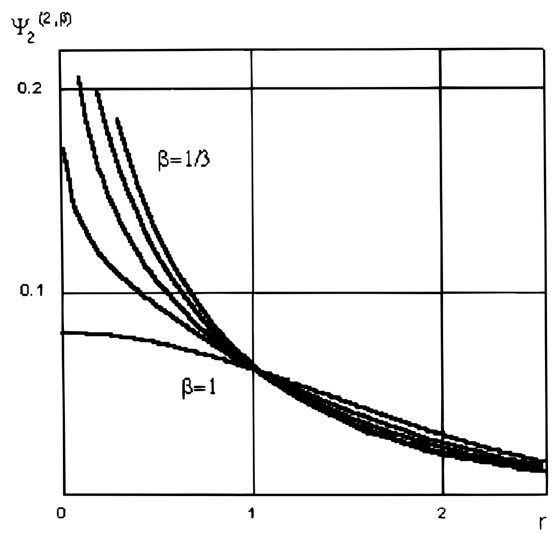

Fig. 2. Two-dimensional ADDs $\Psi_2^{(2,\beta)}(r)$ for the same values of β as in Fig. 1.

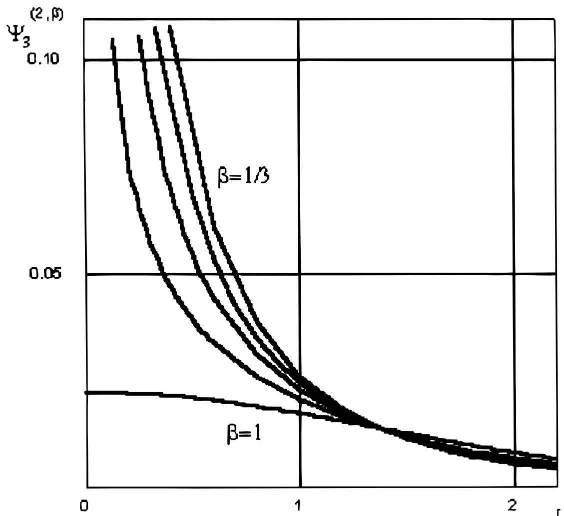

Fig. 3. Three-dimensional ADDs $\Psi_3^{(2,\beta)}(r)$ for the same values of β as in Fig. 1.

6. FOX FUNCTION REPRESENTATION OF ADDs

The Fox function or H-function, also called the generalized G-function or generalized Mellin–Barnes function, is a general function of hypergeometrical type [9, 14, 24]. It represents a rich class of functions which contains functions such as Meijer's G-function, hypergeometric functions, Wright's hypergeometric series, Bessel functions, Mittag–Leffler functions, etc., as special cases.

Let m, n, p, and q be integer numbers such that $0 \le n \le p$ and $1 \le m \le q$. The Fox function of order (m, n, p, q) is defined by the Mellin–Barnes type integral

$$H_{pq}^{mn}\left(z\left|\begin{matrix}(a_1, \alpha_1)\ldots(a_n, \alpha_n) & (a_{n+1}, \alpha_{n+1})\ldots(a_p, \alpha_p)\\(b_1, \beta_1)\ldots(b_m, \beta_m) & (b_{m+1}, \beta_{m+1})\ldots(b_q, \beta_q)\end{matrix}\right.\right) = \frac{1}{2\pi i}\int_C h(s) z^s\, ds$$

where

$$h(s) = \frac{A(s)B(s)}{C(s)D(s)}$$

$$A(s) = \prod_{j=1}^{m} \Gamma(b_j - \beta_j s)$$

$$B(s) = \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)$$

$$C(s) = \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s)$$

$$D(s) = \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)$$

and empty products are interpreted as unity. The parameters $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$ are positive numbers and $a_1, \ldots, a_p, b_1, \ldots, b_q$ are complex numbers satisfying

$$\alpha_j(b_k + \nu) \neq \beta_k(a_j - 1 - \lambda)$$

for $\nu, \lambda = 0, 1, \ldots; k = 1, \ldots, m; j = 1, \ldots, n$. Here C is a contour in the complex s-plane separating the poles in such a way that the poles of $A(s)$ lie to the right and the poles of $B(s)$ lie to the left of the contour.

Let

$$\mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

and

$$\beta = \prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=1}^{q} \beta_j^{-\beta_j}$$

The Fox function is an analytic function of z (i) for every $z \neq 0$ if $\mu > 0$ and (ii) for $0 < |z| < \beta^{-1}$ if $\mu = 0$. In general, the Fox function is multiple valued due to the factor z^s in the integral representation, but it is single valued on the Riemann surface of $\ln z$.

The theorem of residues enables ones to express the Fox function as the infinite series

$$H_{mn}^{pq}(z) = \sum_{j=1}^{m} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{c_{jk} z^{s_{jk}}}{\beta_j}$$

where

$$s_{jk} = (b_j + k)/\beta_j$$

$$c_{jk} = \frac{A_j(s_{jk}) B(s_{jk})}{C(s_{jk}) D(s_{jk})}$$

and

$$A_j(s) = \prod_{l=1, l\neq j}^{m} \Gamma(b_l - \beta_l s)$$

Using these facts together with those cited in Sections 2 and 5, one can easily represent stable distributions and ADDs in terms of Fox functions:

$$\Psi_m^{(\alpha,1)}(r) = g_m^{(\alpha)}(r) = \frac{1}{2}\,(r\sqrt{\pi})^{-m} H_{21}^{11}\left(\frac{2}{r}\middle|\begin{matrix}(1-m/2,\,1/2)(1,\,1/2)\\(1,\,1/\alpha)\end{matrix}\right), \qquad \alpha < 1 \tag{6.1}$$

$$\Psi_m^{(\alpha,1)}(r) = g_m^{(\alpha)}(r) = \frac{2}{\alpha}\,(2\sqrt{\pi})^{-m} H_{12}^{11}\left(\frac{r^2}{4}\middle|\begin{matrix}(1-m/\alpha,\,2/\alpha)\\(0,\,1)(1-m/2,\,1)\end{matrix}\right), \qquad \alpha \leq 1 \tag{6.2}$$

$$g_1^{(\beta,1)}(t) = \frac{1}{\beta}\,t^{-2} H_{11}^{10}\left(\frac{1}{t}\middle|\begin{matrix}(-1,\,1)\\(-1/\beta,\,1/\mathrm{gb})\end{matrix}\right), \qquad \beta < 1$$

$$\Psi_m^{(\alpha,\beta)}(r) = \frac{\beta}{2}\,(4\pi)^{-m/2}\left(\frac{2}{r}\right)^{m+\alpha}$$

$$\times\, H_{32}^{12}\left(\left(\frac{2}{r}\right)^{\beta}\middle|\begin{matrix}(-1,\,1/\alpha)(1-(\alpha+m)/2,\,\beta/2)(1-\alpha/2,\,1/2)\\(0,\,1/\alpha)(-1,\,1/\alpha)\end{matrix}\right),$$

$$\alpha < 1$$

$$\Psi_m^{(\alpha,\beta)}(r) = (\alpha r\sqrt{\pi})^{-m} H_{23}^{21}\left(\frac{r}{2}\middle|\begin{matrix}(1,\,1/\alpha)(1,\,\beta/\alpha)\\(1,\,1/\alpha)(m/2,\,1/2)(1,\,1/2)\end{matrix}\right), \qquad 1 \leq \alpha < 2$$

$$\Psi_m^{(2,\beta)}(r) = (2r\sqrt{\pi})^{-m} H_{12}^{20}\left(\frac{r}{2}\middle|\begin{matrix}(1,\,\beta/2)\\(1,\,1/2)(m/2,\,1/2)\end{matrix}\right) \tag{6.3}$$

7. CONCLUDING REMARKS

The main result of this article is represented by equations (4.6)–(4.8), their solutions (3.5), (3.6) with the properties discussed in Sec. 5 and results of numerical calculations (Sec. 7). But special cases were considered in the works published by different authors in the last decade or so. Let us indicate some of them.

The first supposition about the fractional kind of the equation similar to (4.3) for description of diffusion in a turbulent medium was made in [Monin, 1995]. Weissman and coworkers [Weissman *et al.*, 1989] note that the approximation (5.1) was found by Daniels [Daniels, 1954]. The result (5.2) was obtained in [Tunaley, 1974]. A special version of Eq. (4.8) for

$\alpha = 2$ and $m = 1$ was derived by Balakrishnan [Balakrishnan, 1985] (see also [Barkai, 2000]). The one-dimensional version of Eq. (4.5) coincides with Eq. (2.1) of the work [Wyss, 1986] and with Eq. (27) of the work [Nigmatullin, 1986]. Eq. (4.8) with $\alpha = 2$, (5.2) and (6.3) are considered in [Schneider & Wyss, 1989] (Eqs. (3.1); (3.17) and (2.14) with $k = 0$ correspondingly). Fourier transform (3.3) is in agreement with formula (57) of the work [Afanasiev *et al.*, 1991]. In the one-dimensional case, the distributions (6.1) and (6.2) coincide with symmetrical densities following from Eqs. (2.15) and (2.12) of the work [Schneider, 1986]. For $m = 1$ equation (3.4) corresponds to Eq. (20) of the work [Schlesinger *et al.*, 1982] and in the case of many dimensions it is obtained in [Hilfer, 1995]. Eqs. (13)–(15) of the work [Compte, 1996] are special cases of our equation (4.6) up to notations. Eqs. (38) and (39) from [Compte *et al.*, 1997] are three-dimensional versions of our equations (4.5) and (4.3) correspondingly. Eq. (20) of the work [West *et al.*, 1997] coincides with one-dimensional version of our Eq. (4.7) under conditions $\alpha = 2$ and $\beta = 1 - \beta'$. In the one-dimensional case Eq. (4.4) reduces to Eq. (6.6) of the work [Zaslavsky, 1994] being written for symmetrical diffusion, and Eqs. (3.5)–(3.6) are in agreement with Eqs. (36) and (38) of the work [Kotulski, 1995]. Such agreement takes place with other one-dimensional results obtained in works [Mainardi, 1999, Saichev, 1997] and others.

Notice that the incorrect notation for fractional derivative in [West *et al.*, 1997] (see formulas (19)–(20) in [West *et al.*, 1997]) led the authors to the incorrect conclusion that the case $\beta' < 1$ corresponds to the superdiffusive regime. The assertion that the fractional diffusion equations have solutions in the form of Gaussian density with a rectified variance (formula (20.12.4) of the book [Klimontovich, 1995]) does not correspond to the facts.

APPENDIX: EVALUATION OF a_n

Let X be an m-dimensional random vector with spherically symmetric distribution such that

$$\mathrm{Prob}\{|X| > r\} = \begin{cases} Ar^{-\alpha}, & r > A^{1/\alpha} \\ 1, & r < A^{1/\alpha} \end{cases}$$

Its characteristic function $\varphi(k)$ is of the form

$$\begin{aligned}\varphi_X(k) &= \int_{|x|>A^{1/\alpha}} e^{ik\cdot x} p_x(x)\, d^m x \\ &= 2^{m/2-1}\alpha A\Gamma(m/2)|k|^{\alpha} \int_{|k|A^{1/\alpha}}^{\infty} \xi^{-\alpha-m/2} J_{m/2-1}(\xi)\, d\xi\end{aligned}$$

On integrating by parts, we bring the integral to the form

$$\int_{|k|A^{1/\alpha}}^{\infty} \xi^{-\alpha-m/2} J_{m/2-1}(\xi)\, d\xi$$
$$= \alpha^{-1}(|k|A^{1/\alpha})^{-\alpha-m/2+1} J_{m/2-1}(|k|A^{1/\alpha})$$
$$- \alpha^{-1} \int_{|k|A^{1/\alpha}}^{\infty} \xi^{-\alpha-m/2+1} J_{m/2}(\xi)\, d\xi$$
$$\sim [2^{m/2-1}\alpha A\Gamma(m/2)]^{-\alpha}|k|^{-\alpha} - \alpha^{-1} \int_0^{\infty} \xi^{-\alpha-m/2+1} J_{m/2}(\xi)\, d\xi,$$
$$|k| \to 0$$

Hence,

$$\varphi_X(k) \sim 1 - 2^{m/2-1} A\Gamma(m/2)|k|^{\alpha} \int_0^{\infty} \xi^{-\alpha-m/2+1} J_{m/2}(\xi)\, d\xi$$
$$= 1 - 2^{-\alpha} A \frac{\Gamma(m/2)\Gamma(1-\alpha/2)}{\Gamma((m+\alpha)/2)} |k|^{\alpha}, \qquad |k| \to 0$$

and the characteristic function of the sum $S_n = \Sigma_{i=1}^n X_i$ is

$$\varphi_{S_n}(k) \sim \left[1 - 2^{-\alpha} A \frac{\Gamma(m/2)\Gamma(1-\alpha/2)}{\Gamma((m+\alpha)/2)} |k|^{\alpha}\right]^n, \qquad |k| \to 0$$

Comparing the characteristic function of the normalized vectorial sum $Z_n = s_n/a_n$

$$\varphi_{Z_n}(k) = \varphi_{S_n}(k/a_n), \qquad n \to \infty$$

with the limiting form (10), one finds out that

$$a_n = a_1^{(m)}(\alpha)n^{1/\alpha}, \qquad a_1^{(m)}(\alpha) = \frac{1}{2}\left[A \frac{\Gamma(m/2)\Gamma(1-\alpha/2)}{\Gamma((m+\alpha)/2)}\right]^{1/\alpha}$$

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REFERENCES

V. V. Afanasiev, R. Z. Sagdeev, and G. M. Zaslavsky, (1991). *Chaos* **1**(2), 143.
V. Balakrishnan, (1985). *Physica A* **132**, 569.
E. Barkai, R. Metzler, and J. Klafter, (2000). *Phys. Rev. E* ***61***, 192.

L. N. Bol'shev, V. M. Zolotarev, E. S. Kedrova and M.A. Rybinskaya, (1968). *Theor. Probability Appl.* **15**, 299.

J.-P. Bouchaud, and A. Georges, (1990). *Phys. Rep.* **195**, 127.

A. Compte, (1996). *Phys. Rev. E* **53**, 4191.

A. Compte, D. Jou, and Y. Katayama, (1997). *J. Phys. A: Math. Gen.* **30**, 1023.

H. E. Daniels, (1954). *Ann. Math. Phys.* **25**, 631.

E. F. Fama, and R. Roll. (1968). *J. Amer. Statist. Assoc.* **63**, 817.

W. Feller, (1971). *An Introduction to Probability Theory,* Vol. 2 (Wiley, New York).

C. Fox, (1961). *Trans. Am. Math. Soc.* **98**, 395.

R. Hilfer, (1995). *Fractals* **3**, 211.

D. R. Holt and E. L. Crow, (1973). *J. Res. NBS—B. Math. Sci.* **77B**, 143.

M. B. Isichenko, (1992). *Rev. Mod. Phys.* **64**, 961.

Yu. L. Klimontovich, (1995). *Statistical Theory of Opened Systems*, Kluwer, Dordrecht.

M. Kotulski, (1995). *J. Stat. Phys.* **81**, 771.

F. Mainardi, P. Paradisi, and R. Gorenflo, (1999). In *Econophysics: An Emerging Science*, J. Kertesz and I. Kondor, eds., Kluwer, Dordrecht, in press, pp. 39.

A. M. Mathai and R. K. Saxena, (1973). *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Lect. Not. Math. 348, (Springer Verlag, Heidelberg).

K. S. Miller and B. Ross, (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*, (Wiley, New York).

A. S. Monin, (1955). *Doklady Akad. Nauk SSSR* **105**, 256 (in Russian).

E. W. Montroll and G. H. Weiss, (1965). *J. Math. Phys.* **6**, 167.

R. R. Nigmatullin, (1986). *Phys. Stat. Sol (b)* **133**, 425.

K. B. Oldham and J. Spanier, (1974). *The Fractional Calculus*, (Academic Press, New York).

A. I. Saichev and G. M. Zaslawsky, (1997). *Chaos* **7**(4), 753.

S. G. Samko, A. A. Kilbas, and O. I. Marichev, (1993). *Fractional Integrals and Derivatives—Theory and Applications*, (Gordon and Breach, New York).

W. R. Schneider, (1986). *Lecture Notes in Physics*, Vol. 262, Springer, Berlin, 497.

W. R. Schneider and W. Wyss, (1989). *J. Math. Phys.* **30**, 134.

M. F. Shlesinger, J. Klafter, and Y. M. Wong, (1982). *J. Stat. Phys.* **27**(3).

H. M. Srivastava, K. C. Gupta, and S. P. Goyal, (1982). *The H-Functions of One and Two Variables with Applications*, (South Asian Publishers, New Delhi).

J. K. E. Tunaley, (1974). *J. Stat. Phys.* **11**, 397.

V. V. Uchaikin and V. M. Zolotarev, (1999). *Chance and Stability. Stable Distributions and their Applications*, (VSP, Utrecht, The Netherlands).

H. Weissman, G. H. Weiss, and S. Havlin, (1989). *J. Stat. Phys.* **57**, 301.

B. J. West and W. Deering, (1994). *Phys. Rep.* **246**, 1.

B. J. West, P. Grigolini, R. Metzler, and T. F. Nonnenmacher, (1997). *Phys. Rev. E* **55**, 100.

G. J. Worsdale and J. Roy, (1975). *Statist. Soc. Ser. C: Appl. Statist.* **24**, 123.

W. Wyss, (1986). *J. Math. Phys.* **27**, 2782.

G. M. Zaslavsky, (1994). *Physica D* **76**, 110.

V. M. Zolotarev, (1981). In *Contribution to Probability* (Collection Dedicated to Eugene Lukass), eds. J. Gani and V. K. Rohatgi, Academic Press, London, 283.